

# Conservation laws in the teleparallel theory of gravity

M. Blagojević<sup>1,2</sup> and M. Vasilic<sup>1,\*</sup>

<sup>1</sup>*Institute of Physics, 11001 Belgrade, P. O. Box 57, Yugoslavia*

<sup>2</sup>*PINT, 6001 Koper, P. O. Box 327, Slovenia*

## Abstract

We study the conservation laws associated with the asymptotic Poincaré symmetry of spacetime in the general teleparallel theory of gravity. Demanding that the canonical Poincaré generators have well defined functional derivatives in a properly defined phase space, we obtain the improved form of the generators, containing certain surface terms. These terms are shown to represent the values of the related conserved charges: energy-momentum and angular momentum.

## I. INTRODUCTION

A field theory is defined by both the field equations and boundary conditions. In contrast to the usual flat-space field theories, the boundary conditions in gravitational theories define the asymptotic structure of spacetime. The concept of asymptotic or boundary symmetry is of fundamental importance for understanding the conservation laws in gravity; it is defined by the gauge transformations that leave a chosen set of boundary conditions invariant. The asymptotic symmetry has a very clear dynamical interpretation: the symmetry of the action breaks down to the symmetry of boundary conditions, which plays the role of the physical symmetry and defines the corresponding conservation laws. A consistent picture of the gravitational energy and other conserved quantities in general relativity (GR) emerged only after the role of boundary conditions and their symmetries had been fully recognized [1–3].

The Poincaré gauge theory [4–6] (PGT) is a natural extension of the gauge principle to spacetime symmetries, and represents a viable alternative to general relativity (GR) (for more general attempts see Ref. [7]). A particularly interesting limit of PGT is given by the teleparallel geometry [8–11]. The teleparallel description of gravity has been a promising alternative to GR until the work of Kopczyński [12], who found a hidden gauge symmetry which prevents the torsion from being completely determined by the field equations, and concluded that the theory is inconsistent. Possible consequences of this conclusion have been further discussed by Müller-Hoisson and Nitsch [13]. Nester [14] improved the arguments by showing that the unpredictable behavior of torsion occurs only for some very special solutions (see also Refs. [15]).

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\*Email addresses: mb@phy.bg.ac.yu and mvasilic@phy.bg.ac.yu

The canonical analysis of the teleparallel formulation of GR [16] (see also Ref. [17]) is an important step towards clarifying the gauge structure of the teleparallel theory. In this case, the presence of non-dynamical torsion components is shown to be not a sign of an inconsistency, but a consequence of the constraint structure of the theory. The undetermined torsion components appear as a consequence of extra gauge symmetries (with respect to which the torsion tensor is not a covariant object). In the general teleparallel theory, the influence of extra gauge symmetries on the existence of a consistent coupling with matter is presently not completely clear. We shall assume that matter coupling respects all extra gauge symmetries of the gravitational sector, if they exist.

Hecht et. al. [18] investigated the initial value problem of the teleparallel form of GR, and concluded that it is well defined if the undetermined velocities are dropped out from the set of dynamical velocities. The problem has not been analyzed for more general teleparallel theories, but the results of Hecht et. al. [19], related to  $T^2$  theories in  $U_4$ , are encouraging. Among various successful applications of the teleparallel approach, one should mention a pure tensorial proof of the positivity of energy in GR [20], a transparent introduction of Ashtekar's complex variables [21], and a formulation of a five-dimensional teleparallel equivalent of the Kaluza-Klein theory [22]. Although quantum properties of PGT are in general not so attractive [23,24], the related behavior in the specific case of the teleparallel theory might be better [25,26], and should be further explored.

At the classical level, further progress has been made by carrying out an explicit construction of the generators of all gauge symmetries of the general teleparallel theory [27]. In the present paper, we continue this investigation by studying the important relation between asymptotic symmetries and conserved charges for isolated gravitating systems. Assuming that the symmetry in the asymptotic region is given by the global Poincaré symmetry, we shall use the Regge-Teitelboim approach to find out the form of the improved canonical generators [3,28]. The method is based on the fact that the canonical generators act on dynamical variables via Poisson brackets, which implies that they should have well defined functional derivatives. The global Poincaré generators do not satisfy this requirement unless we redefine them by adding certain surface terms, which are shown to represent the conserved values of the energy-momentum and angular momentum.

Basic gravitational variables in PGT are the tetrad field  $b^k_\mu$  and the Lorentz connection  $A^{ij}_\mu$ , and their field strengths are geometrically identified with the torsion and the curvature, respectively:  $T^k_{\mu\nu} = \partial_\mu b^k_\nu - A^k_{s\mu} b^s_\nu - (\mu \leftrightarrow \nu)$ ,  $R^{ij}_{\mu\nu} = \partial_\mu A^{ij}_\nu - A^i_{s\mu} A^{sj}_\nu - (\mu \leftrightarrow \nu)$ . General geometric structure of PGT is described by the Riemann-Cartan space  $U_4$ , possessing metric (or tetrad) and metric compatible connection. The teleparallel or Weitzenböck geometry  $T_4$  can be formulated as a special limit of PGT, characterized by a metric compatible connection possessing non-vanishing torsion, while the curvature is restricted to vanish (see also [29]):

$$R^{ij}_{\mu\nu}(A) = 0. \quad (1.1)$$

Teleparallel theories describe the dynamical content of spacetime by a class of Lagrangians quadratic in the torsion:

$$\begin{aligned} \mathcal{L} &= b(\mathcal{L}_T + \mathcal{L}_M) + \lambda_{ij}{}^{\mu\nu} R^{ij}_{\mu\nu}, \\ \mathcal{L}_T &= a(AT_{ijk}T^{ijk} + BT_{ijk}T^{jik} + CT_kT^k) \equiv \beta_{ijk}(T)T^{ijk}. \end{aligned} \quad (1.2)$$

Here,  $\mathcal{L}_M$  is the matter Lagrangian, the Lagrange multipliers  $\lambda_{ij}{}^{\mu\nu}$  ensure the teleparallelism condition (1.1),  $A, B$  and  $C$  are free parameters [30],  $a = 1/2\kappa$  ( $\kappa$  is Einstein's gravitational

constant), and  $T_k = T^m_{mk}$ . The gravitational field equations, obtained by varying  $\mathcal{L}$  with respect to  $b^i_\mu$ ,  $A^{ij}_\mu$  and  $\lambda_{ij}{}^{\mu\nu}$ , have the form:

$$4\nabla_\rho(b\beta_i{}^{\mu\rho}) - 4b\beta^{nm\mu}T_{nmi} + h_i{}^\mu b\mathcal{L}_T = \tau^\mu_i, \quad (1.3a)$$

$$4\nabla_\rho\lambda_{ij}{}^{\mu\rho} - 8b\beta_{[ij]}{}^\mu = \sigma^\mu_{ij}, \quad (1.3b)$$

$$R^{ij}{}_{\mu\nu} = 0, \quad (1.3c)$$

where  $\tau^\mu_i$  and  $\sigma^\mu_{ij}$  are the energy-momentum and spin currents of matter fields, respectively. If the gravitational sector of the theory possesses extra gauge symmetries, the matter coupling is assumed to respect them. The physical interpretation of the teleparallel theories is based on the observation that there exists a one-parameter family of the teleparallel Lagrangians (1.2), defined by the condition *i*)  $2A + B + C = 0$ ,  $C = -1$ , which represents a viable gravitational theory for macroscopic matter, observationally indistinguishable from GR [9–11]. For the parameter value *ii*)  $B = 1/2$ , the gravitational part of (1.2) coincides, modulo a four-divergence, with the Hilbert-Einstein Lagrangian, and yields the teleparallel form of GR,  $\text{GR}_\parallel$ .

The layout of the paper is as follows. In Sec. II, we construct the generators of the asymptotic (global) Poincaré symmetry from the corresponding local expressions [27], and introduce an appropriate asymptotic structure of the phase space in which these generators act. Then, in Sec. III, we impose the requirement that the asymptotic Poincaré generators have well defined functional derivatives. This leads to the appearance of certain surface terms in the generators, which define the values of the related charges: energy-momentum and angular momentum of the gravitating system. In the next section, we discuss the conservation laws of these charges, which are associated with the asymptotic symmetry of spacetime. In Sec. V, we transform our expressions for the conserved charges into the Lagrangian form, and compare them with the known GR results. Finally, Sec. VI is devoted to concluding remarks, while some technical details are presented in the Appendices.

Our conventions are the same as in Refs. [16,17,27,28]: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of both alphabets ( $a, b, c, \dots; \alpha, \beta, \gamma, \dots$ ) run over 1, 2, 3, and the middle alphabet letters ( $i, j, k, \dots; \mu, \nu, \lambda, \dots$ ) run over 0, 1, 2, 3;  $\eta_{ij} = (+, -, -, -)$ , and  $\varepsilon^{ijkl}$  is completely antisymmetric symbol normalized by  $\varepsilon^{0123} = +1$ .

## II. ASYMPTOTIC SYMMETRY OF SPACETIME

We begin our considerations by discussing the asymptotic structure of spacetime and the related form of the symmetry generators in the teleparallel theory. Our attention will be limited to gravitating systems which are characterized by the global Poincaré symmetry at large distances.

### A. Poincaré invariance in the asymptotic region

The generators of local Poincaré transformations in the teleparallel theory of gravity without matter are constructed in Ref. [27] (Appendix A). The global Poincaré transfor-

mations of fields and momenta are obtained from the corresponding local transformations, given in Eqs. (2.3) and (5.2) of Ref. [27], by the replacements

$$\omega^{ij}(x) \rightarrow \varepsilon^{ij}, \quad \xi^\mu(x) \rightarrow \varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu, \quad (2.1)$$

where  $\varepsilon^{ij}, \varepsilon^\mu$  are constants, and  $\varepsilon^\mu{}_\nu = \delta^\mu_i \eta_{j\nu} \varepsilon^{ij}$ . We use the usual convention according to which indices of quantities related to the asymptotic spacetime are raised and lowered by the Minkowski metric, while the transition between local Lorentz and coordinate basis is realized with the help of the Kronecker symbol. In an analogous manner, the global Poincaré generators are obtained from the corresponding local expressions (A4):

$$G = -\varepsilon^\mu P_\mu + \frac{1}{2} \varepsilon^{ij} M_{ij}, \quad (2.2a)$$

where

$$\begin{aligned} P_\mu &= \int d^3x \mathcal{P}_\mu, & M_{\mu\nu} &= \int d^3x \mathcal{M}_{\mu\nu}, \\ \mathcal{M}_{\alpha\beta} &= -S_{\alpha\beta} + x_\alpha \mathcal{P}_\beta - x_\beta \mathcal{P}_\alpha \\ \mathcal{M}_{0\beta} &= -S_{0\beta} + x_0 \mathcal{P}_\beta - x_\beta \mathcal{P}_0 + b^k{}_{\beta\pi k}{}^0 + \frac{1}{2} A^{ij}{}_{\beta\pi ij}{}^0 - \frac{1}{2} \lambda_{ij}{}^{0\gamma} \pi^{ij}{}_{\beta\gamma}. \end{aligned} \quad (2.2b)$$

The quantities  $\mathcal{P}_\mu$  and  $S_{\mu\nu} = \delta_\mu^i \delta_\nu^j S_{ij}$  are defined in Appendix A.

## B. Boundary conditions

We shall be concerned here with *finite* gravitational sources, characterized by matter fields which decrease sufficiently fast at large distances, so that their contribution to surface integrals vanishes. Consequently, the matter field contribution to the symmetry generators can be effectively ignored. In that case, we can assume that spacetime is *asymptotically flat*, i.e. that the following two conditions are fulfilled [31,32]:

- a) there exists a coordinate system in which the metric  $g_{\mu\nu}$  of spacetime becomes Minkowskian at large distances:  $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}_1$ , where  $\mathcal{O}_n = \mathcal{O}(r^{-n})$  denotes a term which decreases like  $r^{-n}$  or faster for large  $r$ , and  $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ ;
- b) the Lorentz field strength defines the absolute parallelism for large  $r$ :  $R^{ij}{}_{\mu\nu} = \mathcal{O}_{2+\alpha}$ , with  $\alpha > 0$ .

The first conditions is self evident, and the second one is trivially satisfied in the teleparallel theory, where  $R^{ij}{}_{\mu\nu}(A) = 0$ . The vanishing of the curvature means that the Lorentz gauge potential  $A^{ij}{}_\mu$  is a pure gauge, hence it can be transformed to zero by a suitable local Lorentz transformation. Therefore, we can adopt the following asymptotic behavior of the translational and Lorentz gauge fields:

$$b^i{}_\mu = \delta_\mu^i + \mathcal{O}_1, \quad A^{ij}{}_\mu = \hat{\mathcal{O}}, \quad (2.3a)$$

where  $\hat{\mathcal{O}}$  denotes a term with an arbitrarily fast asymptotic decrease.

There is one more Lagrangian variable in the teleparallel theory, the Lagrange multiplier  $\lambda_{ij}{}^{\mu\nu}$ , which is not directly related to the above geometric conditions a) and b). Its

asymptotic behavior can be determined with the help of the second field equation, leading to

$$\lambda_{ij}{}^{\mu\nu} = \text{const.} + \mathcal{O}_1. \quad (2.3b)$$

The vacuum values of the fields,  $b^i{}_\mu = \delta^i_\mu$ ,  $A^{ij}{}_\mu = 0$  and  $\lambda_{ij}{}^{\mu\nu} = \text{const.}$ , are taken to be invariant under the action of the global Poincaré group. Demanding that the asymptotic conditions (2.3) be invariant under the global Poincaré transformations, we obtain the following conditions on the field derivatives:

$$\begin{aligned} b^k{}_{\mu,\nu} &= \mathcal{O}_2, & b^k{}_{\mu,\nu\rho} &= \mathcal{O}_3, \\ \lambda_{ij}{}^{\mu\nu}{}_{,\rho} &= \mathcal{O}_2, & \lambda_{ij}{}^{\mu\nu}{}_{,\rho\lambda} &= \mathcal{O}_3. \end{aligned} \quad (2.4)$$

The above relations impose serious restrictions on the gravitational field in the asymptotic region, and define an *isolated* gravitational system (characterized, in particular, by the absence of gravitational waves).

The asymptotic behavior (2.3) and (2.4) is compatible not only with global Poincaré transformations, but also with a restricted set of local Poincaré transformations, whose gauge parameters decrease sufficiently fast for large  $r$ .

In addition to the asymptotic conditions (2.3) and (2.4), we shall adopt the principle that *all the expressions that vanish on shell have an arbitrarily fast asymptotic decrease*, as no solution of the equations of motion is thereby lost. In particular, the constraints of the theory are assumed to decrease arbitrarily fast, and consequently, all volume integrals defining the Poincaré generators (2.2) are convergent.

The asymptotic behavior of momentum variables is determined using the asymptotics for the fields and the relation  $\pi - \partial\mathcal{L}/\partial\dot{\varphi} = \hat{\mathcal{O}}$ , in accordance with the above principle. Thus, we find

$$\begin{aligned} \pi_i{}^0, \pi_{ij}{}^0, \pi^{ij}{}_{\mu\nu} &= \hat{\mathcal{O}}, \\ \pi_i{}^\alpha &= \mathcal{O}_2, \\ \pi_{ij}{}^\alpha &= 4\lambda_{ij}{}^{0\alpha} + \hat{\mathcal{O}}. \end{aligned} \quad (2.5)$$

In a similar manner, we can determine the asymptotic behavior of the Hamiltonian multipliers (for instance,  $\dot{A}^{ij}{}_0 = u^{ij}_0$  implies  $u^{ij}_0 = \hat{\mathcal{O}}$ , etc.).

Now, we wish to check whether the global Poincaré generators (2.2) are well defined in the phase space characterized by the asymptotic properties (2.3)-(2.5).

### III. IMPROVING THE POINCARÉ GENERATORS

In the Hamiltonian theory, the generators of symmetry transformations act on dynamical variables via the Poisson bracket operation, which is defined in terms of functional derivatives. A functional

$$F[\varphi, \pi] = \int d^3x f(\varphi(x), \partial_\mu \varphi(x), \pi(x), \partial_\nu \pi(x))$$

has well defined functional derivatives if its variation can be written in the form

$$\delta F = \int d^3x [A(x)\delta\varphi(x) + B(x)\delta\pi(x)], \quad (3.1)$$

where terms  $\delta\varphi_{,\mu}$  and  $\delta\pi_{,\nu}$  are absent.

The global Poincaré generators (2.2) do not satisfy these requirements, as we shall see. This will lead us to improve their form by adding certain surface terms, which turn out to define energy-momentum and angular momentum of the gravitating system [3,28].

### A. Spatial translation

The variation of the spatial translation generator  $P_\alpha$ , given by Eq. (A4c), yields

$$\begin{aligned} \delta P_\alpha &= \int d^3x \delta \mathcal{P}_\alpha, \\ \delta \mathcal{P}_\alpha &= \delta \bar{\mathcal{H}}_\alpha - \frac{1}{2} A^{ij}_\alpha \delta \bar{\mathcal{H}}_{ij} + 2\lambda_{ij}{}^{0\beta} \delta \bar{\mathcal{H}}^{ij}_{\alpha\beta} + \pi_k{}^0 \delta b^k_{0,\alpha} \\ &\quad + \frac{1}{2} \pi_{ij}{}^0 \delta A^{ij}_{0,\alpha} - \frac{1}{4} \lambda_{ij}{}^{\beta\gamma} \delta \pi^{ij}_{\beta\gamma,\alpha} - \frac{1}{2} \delta (\lambda_{ij}{}^{\beta\gamma} \pi^{ij}_{\alpha\beta})_{,\gamma} + R. \end{aligned} \quad (3.2a)$$

Here, all terms that contain unwanted variations  $\delta\pi_{,\alpha}$  or  $\delta\varphi_{,\alpha}$  are written explicitly, while those that have the correct, regular form (3.1) are denoted by  $R$ . A simple formula  $\pi_k{}^0 \delta b^k_{0,\alpha} = (\pi_k{}^0 \delta b^k_0)_{,\alpha} + R$  allows us to conclude that  $\pi_k{}^0 \delta b^k_{0,\alpha} = \partial \hat{\mathcal{O}} + R$ , where we used  $\pi_k{}^0 = \hat{\mathcal{O}}$ , according to the asymptotic conditions (2.5). Now, we apply the same reasoning to the terms proportional to  $\pi_{ij}{}^0$ ,  $\pi^{ij}_{\alpha\beta}$ ,  $\pi^{ij}_{0\beta}$  (present in  $\bar{\mathcal{H}}^{ij}_{0\beta}$ ) and  $A^{ij}_\mu$ , and find

$$\delta \mathcal{P}_\alpha = \delta \bar{\mathcal{H}}_\alpha + \partial \hat{\mathcal{O}} + R. \quad (3.2b)$$

Using the explicit form of  $\bar{\mathcal{H}}_\alpha$ , equation (A1c), we obtain the result

$$\begin{aligned} \delta \mathcal{P}_\alpha &= -\delta (b^i{}_\alpha \pi_i{}^\gamma)_{,\gamma} + (\pi_i{}^\gamma \delta b^i{}_\gamma)_{,\alpha} + \partial \hat{\mathcal{O}} + R \\ &= -\delta (b^i{}_\alpha \pi_i{}^\gamma)_{,\gamma} + \partial \mathcal{O}_3 + R, \end{aligned} \quad (3.2c)$$

where the last equality follows from  $\pi_i{}^\gamma \delta b^i{}_\gamma = \mathcal{O}_3$ . As a consequence, the variation of the spatial translation generator  $P_\alpha$  can be written in the simple form

$$\begin{aligned} \delta P_\alpha &= -\delta E_\alpha + R, \\ E_\alpha &\equiv \oint dS_\gamma (b^k{}_\alpha \pi_k{}^\gamma), \end{aligned} \quad (3.3)$$

where  $E_\alpha$  is defined as a surface integral over the boundary of the three-dimensional space. This result allows us to redefine the translation generator  $P_\alpha$ ,

$$P_\alpha \rightarrow \tilde{P}_\alpha = P_\alpha + E_\alpha, \quad (3.4)$$

so that the new, improved expression  $\tilde{P}_\alpha$  has *well defined functional derivatives*.

The surface integral for  $E_\alpha$  is *finite* since  $b^k{}_\alpha \pi_k{}^\gamma = \mathcal{O}_2$ , in view of the asymptotic conditions (2.5). While the old generator vanishes on shell (as an integral of a linear combination of constraints),  $\tilde{P}$  does not — its on-shell value is  $E_\alpha$ . Since  $\tilde{P}_\alpha$  is the generator of the asymptotic spatial translations, we expect that  $E_\alpha$  will be the value of the related conserved charge — linear momentum; this will be proved in Sec. 4.

## B. Time translation

Similar procedure can be applied to the time translation generator  $P_0$ :

$$\begin{aligned}\delta P_0 &= \int d^3x \delta \mathcal{P}_0, \\ \delta \mathcal{P}_0 &= \delta \mathcal{H}_T - \delta(b^k_{\phantom{k}0} \pi_k^\gamma)_{,\gamma} + \partial \hat{\mathcal{O}},\end{aligned}\tag{3.5a}$$

where we used Eq. (A4c) for  $\mathcal{P}_0$ , and the adopted asymptotic conditions for  $A^{ij}_0$  and  $\pi^{ij}_{0\beta}$ . Since  $\mathcal{H}_T$  does not depend on the derivatives of momenta (on shell), as shown in Appendix C of Ref. [27], we can write

$$\begin{aligned}\delta \mathcal{H}_T &= \frac{\partial \mathcal{H}_T}{\partial b^k_{\phantom{k}\mu,\alpha}} \delta b^k_{\phantom{k}\mu,\alpha} + \frac{1}{2} \frac{\partial \mathcal{H}_T}{\partial A^{ij}_{\phantom{ij}\mu,\alpha}} \delta A^{ij}_{\phantom{ij}\mu,\alpha} + R \\ &\approx -\frac{\partial \mathcal{L}}{\partial b^k_{\phantom{k}\mu,\alpha}} \delta b^k_{\phantom{k}\mu,\alpha} - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial A^{ij}_{\phantom{ij}\mu,\alpha}} \delta A^{ij}_{\phantom{ij}\mu,\alpha} + R.\end{aligned}$$

The second term has the form  $\partial \hat{\mathcal{O}} + R$ , so that

$$\delta \mathcal{H}_T \approx -\partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial b^k_{\phantom{k}\mu,\alpha}} \delta b^k_{\phantom{k}\mu} \right) + \partial \hat{\mathcal{O}} + R = \partial \mathcal{O}_3 + R,\tag{3.5b}$$

and we find that

$$\delta \mathcal{P}_0 = -\delta(b^k_{\phantom{k}0} \pi_k^\gamma)_{,\gamma} + \partial \mathcal{O}_3 + R.$$

Hence,

$$\begin{aligned}\delta P_0 &= -\delta E_0 + R, \\ E_0 &= \oint dS_\gamma (b^k_{\phantom{k}0} \pi_k^\gamma).\end{aligned}\tag{3.6}$$

The improved time translation generator

$$\tilde{P}_0 = P_0 + E_0\tag{3.7}$$

has well defined functional derivatives, and the surface term  $E_0$  is finite on account on the adopted asymptotics. As we shall see, the on-shell value  $E_0$  of the time translation generator represents the energy of the gravitating system.

Expressions (3.6) and (3.3) for the energy and momentum can be written in a Lorentz covariant form:

$$E_\mu = \oint dS_\gamma (b^k_{\phantom{k}\mu} \pi_k^\gamma).\tag{3.8}$$

It is interesting to observe that the value of the energy  $E_0$  can be calculated from the formula  $E_0 = \int d^3x \mathcal{H}_T$ . Indeed,

$$\int d^3x \mathcal{H}_T \approx \int d^3x \partial_\gamma \bar{D}^\gamma = \oint dS_\gamma \bar{D}^\gamma = E_0,$$

since  $\bar{D}^\gamma = b^k_{\phantom{k}0} \pi_k^\gamma + \hat{\mathcal{O}}$ .

### C. Rotation

Next, we want to check if the rotation generator  $M_{\alpha\beta}$  has well defined functional derivatives:

$$\begin{aligned}\delta M_{\alpha\beta} &= \int d^3x \delta \mathcal{M}_{\alpha\beta}, \\ \delta \mathcal{M}_{\alpha\beta} &= x_\alpha \delta \mathcal{P}_\beta - x_\beta \delta \mathcal{P}_\alpha + \delta \pi_{\alpha\beta}{}^\gamma{}_{,\gamma} + R,\end{aligned}\tag{3.9}$$

where  $\pi_{\mu\nu}{}^\rho = \delta_\mu^i \delta_\nu^j \pi_{ij}{}^\rho$ . Using the expression (3.2c) for  $\delta \mathcal{P}_\alpha$ , one finds

$$\begin{aligned}\delta M_{\alpha\beta} &= -\delta E_{\alpha\beta} + R, \\ E_{\alpha\beta} &= \oint dS_\gamma \left[ x_\alpha (b^k{}_\beta \pi_k{}^\gamma) - x_\beta (b^k{}_\alpha \pi_k{}^\gamma) - \pi_{\alpha\beta}{}^\gamma \right].\end{aligned}\tag{3.10}$$

In the course of obtaining the above expression, the term  $\oint dS_{[\alpha} x_{\beta]} \phi$  ( $\phi \equiv \pi_k{}^\gamma \delta b^k{}_\gamma$ ) has been discarded as a consequence of  $dS_\alpha \propto x_\alpha$  on the integration sphere. The corresponding improved rotation generator is given by

$$\tilde{M}_{\alpha\beta} \equiv M_{\alpha\beta} + E_{\alpha\beta}.\tag{3.11}$$

Although  $\tilde{M}_{\alpha\beta}$  has well defined functional derivatives, the assumed asymptotics (2.3)-(2.5) does not ensure the finiteness of  $E_{\alpha\beta}$  owing to the presence of  $\mathcal{O}_1$  terms. Note, however, that the actual asymptotics is refined by the principle of arbitrarily fast decrease of all on-shell vanishing expressions. Thus, we can use the constraints  $\bar{\mathcal{H}}_\alpha$  and  $\bar{\mathcal{H}}_{ij}$ , in the lowest order in  $r^{-1}$ , to conclude that

$$\pi_\alpha{}^\beta{}_{,\beta} = \mathcal{O}_4, \quad 2\pi_{[\alpha\beta]} + \pi_{\alpha\beta}{}^\gamma{}_{,\gamma} = \mathcal{O}_3,\tag{3.12}$$

where  $\pi_{\mu\nu} = \delta_\mu^i \eta_{\nu\rho} \pi_i{}^\rho$ . As a consequence, the angular momentum density decreases like  $\mathcal{O}_3$ :

$$\left( 2x_{[\alpha} \pi_{\beta]}{}^\gamma - \pi_{\alpha\beta}{}^\gamma \right)_{,\gamma} = \mathcal{O}_3.\tag{3.13}$$

As all variables in the theory are assumed to have asymptotically polynomial behaviour in  $r^{-1}$ , the integrand of  $E_{\alpha\beta}$  must essentially be of an  $\mathcal{O}_2$  type to agree with the above constraint. The possible  $\mathcal{O}_1$  terms are divergenceless, and do not contribute to the corresponding surface integral (as shown in Appendix B). This ensures the finiteness of the rotation generator.

### D. Boost

By varying the boost generator (2.2b), we find:

$$\begin{aligned}\delta M_{0\beta} &= \int d^3x \delta \mathcal{M}_{0\beta}, \\ \delta \mathcal{M}_{0\beta} &= x_0 \delta \mathcal{P}_\beta - x_\beta \delta \mathcal{P}_0 + \delta \pi_{0\beta}{}^\gamma{}_{,\gamma} + R,\end{aligned}\tag{3.14}$$

in analogy with (3.9). Here, we need to calculate  $\delta \mathcal{P}_0$ , equation (3.5a), up to terms  $\partial \mathcal{O}_4$ . A simple calculation gives

$$\delta\mathcal{M}_{0\beta} = \delta\left(\pi_{0\beta}{}^\gamma - x_0 b^k{}_\beta \pi_k{}^\gamma + x_\beta b^k{}_0 \pi_k{}^\gamma\right)_{,\gamma} + (x_\beta X^\gamma)_{,\gamma} + \partial\mathcal{O}_3 + R, \quad (3.15)$$

where the term

$$X^\gamma \equiv \frac{\partial\mathcal{L}}{\partial b^k{}_{\mu,\gamma}} \delta b^k{}_\mu$$

is not a total variation, nor does it vanish on account of the asymptotic conditions (2.3)–(2.5). To get rid of this unwanted term, we shall further restrict the phase space by adopting the following *parity conditions*:

$$b_{i\mu} = \eta_{i\mu} + \frac{p_{i\mu}(\mathbf{n})}{r} + \frac{q_{i\mu}(t, \mathbf{n})}{r^2} + \mathcal{O}_3, \quad (3.16a)$$

where  $\mathbf{n} = \mathbf{x}/r$  is the three-dimensional unit vector, and

$$\begin{aligned} p_{i\mu}(\mathbf{n}) &= p_{i\mu}(-\mathbf{n}), \\ q_{i\mu} &= q_{i\mu}^{(1)}(\mathbf{n}) + t q_{i\mu}^{(2)}(\mathbf{n}), \quad q_{i\mu}^{(2)}(\mathbf{n}) = -q_{i\mu}^{(2)}(-\mathbf{n}). \end{aligned} \quad (3.16b)$$

Time independence of  $p_{i\mu}$ , and linear time dependence of  $q_{i\mu}$  are a consequence of the required invariance of asymptotic conditions under the global Poincaré transformations. It is straightforward to verify that the *refined asymptotics ensures the vanishing of the unwanted  $X$  term in (3.15)*. Therefore, the improved boost generator, with well defined functional derivatives, has the form

$$\begin{aligned} \tilde{M}_{0\beta} &\equiv M_{0\beta} + E_{0\beta}, \\ E_{0\beta} &= \oint dS_\gamma \left[ x_0 (b^k{}_\beta \pi_k{}^\gamma) - x_\beta (b^k{}_0 \pi_k{}^\gamma) - \pi_{0\beta}{}^\gamma \right]. \end{aligned} \quad (3.17)$$

It remains to be shown that the adopted asymptotics ensures the finiteness of  $E_{0\beta}$ . That this is indeed true can be seen by analyzing the constraints  $\bar{\mathcal{H}}_\perp$  and  $\bar{\mathcal{H}}_{ij}$  at spatial infinity. Using the needed formulas from Appendix A, one finds:

$$\pi_0{}^\beta{}_{,\beta} = \mathcal{O}_4, \quad \pi_{0\beta} + \pi_{0\beta}{}^\gamma{}_{,\gamma} = \mathcal{O}_3. \quad (3.18)$$

One should observe that this result holds in any teleparallel theory; indeed, the relation  $\hat{\mathcal{H}}_T \equiv \mathcal{H}_T - \partial_\alpha \bar{D}^\alpha \approx 0$  implies  $\pi_0{}^\beta{}_{,\beta} \approx \mathcal{H}_T + \mathcal{O}_4 \approx \pi_A \dot{\varphi}^A - \mathcal{L} + \mathcal{O}_4 = \mathcal{O}_4$ . It is now easy to verify that the boost density decreases like  $\mathcal{O}_3$  for large  $r$ . Then, the arguments given in Appendix B lead us to conclude that  $E_{0\beta}$  is finite.

The improved boost generator  $\tilde{M}_{0\beta}$  is a well defined functional on the phase space defined by the refined asymptotic conditions (2.3)–(2.5) and (3.16). The importance of suitably chosen parity conditions for a proper treatment of the angular momentum has been clearly recognized in the past [3,33].

Using the Lorentz 4-notation, we can write:

$$\begin{aligned} \tilde{M}_{\mu\nu} &\equiv M_{\mu\nu} + E_{\mu\nu}, \\ E_{\mu\nu} &= \oint dS_\gamma \left[ x_\mu (b^k{}_\nu \pi_k{}^\gamma) - x_\nu (b^k{}_\mu \pi_k{}^\gamma) - \pi_{\mu\nu}{}^\gamma \right]. \end{aligned} \quad (3.19)$$

#### IV. CONSERVED QUANTITIES

In the preceding section, we obtained the improved Poincaré generators  $(\tilde{P}_\mu, \tilde{M}_{\mu\nu})$ . Their action on the fields and momenta is the same as before, since surface terms act trivially on local quantities. Explicit transformations are obtained by the replacement (2.1) in the local expressions of Ref. [27]. Once we know the generators  $\tilde{P}_\mu$  and  $\tilde{M}_{\mu\nu}$  have the standard Poincaré action on the whole phase space, we deduce their algebra to be that of the Poincaré group:

$$\begin{aligned}\{\tilde{P}_\mu, \tilde{P}_\nu\} &= 0, \\ \{\tilde{P}_\mu, \tilde{M}_{\nu\lambda}\} &= \eta_{\mu\nu}\tilde{P}_\lambda - \eta_{\mu\lambda}\tilde{P}_\nu, \\ \{\tilde{M}_{\mu\nu}, \tilde{M}_{\lambda\rho}\} &= \eta_{\mu\rho}\tilde{M}_{\nu\lambda} - \eta_{\mu\lambda}\tilde{M}_{\nu\rho} - (\mu \leftrightarrow \nu).\end{aligned}\tag{4.1}$$

The line of reasoning that leads to the above result does not guarantee the strong equalities in (4.1). They are rather equalities up to trivial generators, such as squares of constraints and surface terms. In fact, the latter are not expected to appear in (4.1) as a consequence of the result of Ref. [34] that the Poisson bracket of two well defined generators is necessarily a well defined generator. This is of particular importance for the existence of conserved quantities in the theory. In what follows, we shall explicitly verify the absence of these surface terms, and prove the conservation of all the symmetry generators.

The result (4.1) is an expression of the asymptotic Poincaré invariance of the theory. In what follows, our task will be to show that this symmetry implies, as usual, the existence of conserved charges.

The general method for constructing the generators of local symmetries in the Hamiltonian approach has been developed by Castellani [35]. A slight modification of that method can be applied to study global symmetries. One can show that necessary and sufficient conditions for a phase-space functional  $G[\varphi, \pi, t]$  to be a generator of global symmetries take the form:

$$\{G, \tilde{H}\} + \frac{\partial G}{\partial t} = C_{PFC},\tag{4.2a}$$

$$\{G, \phi_m\} \approx 0,\tag{4.2b}$$

where  $\tilde{H}$  is the improved Hamiltonian,  $C_{PFC}$  is a primary first class constraint,  $\phi_m \approx 0$  are all the constraints in the theory, and as before, the equality sign denotes an equality up to surface terms and squares of constraints. The improved Poincaré generators  $\tilde{P}_\mu, \tilde{M}_{\mu\nu}$  are easily seen to satisfy (4.2b), as they are given, up to surface terms (whose action on local quantities is trivial), by volume integrals of first-class constraints. Having in mind that the improved Hamiltonian  $\tilde{H}$  equals  $\tilde{P}_0$ , one can verify that the condition (4.2a) is also satisfied, as a consequence of the part of the algebra (4.1) involving  $\tilde{P}_0$ . The condition (4.2a) is related to conservation laws; this is clearly seen if we rewrite it as a weak equality

$$\frac{dG}{dt} \equiv \{G, \tilde{P}_0\} + \frac{\partial G}{\partial t} \approx S,\tag{4.3}$$

where  $S$  denotes possible surface terms. We see that the generator  $G$  is conserved only if these surface terms are absent. In what follows, we shall explicitly evaluate  $dG/dt$  for each of the generators  $G = \tilde{P}_\mu, \tilde{M}_{\mu\nu}$ , and check their conservation.

1. Let us begin with the energy. First, we note that  $\tilde{P}_0$ , being a well defined functional, must commute with itself:

$$\{\tilde{P}_0, \tilde{P}_0\} = 0.$$

Furthermore,

$$\frac{\partial \tilde{P}_0}{\partial t} = \frac{\partial H_T}{\partial t} = C_{PFC} \approx 0,$$

since the only explicit time dependence of the total Hamiltonian is due to the arbitrary multipliers, which are always multiplied by the primary first class constraints. Therefore,

$$\frac{d\tilde{P}_0}{dt} \approx \frac{dE_0}{dt} \approx 0, \quad (4.4)$$

and we see that the surface term  $E_0$ , representing the on-shell value of the energy, is a conserved quantity.

2. The linear momentum and the spatial angular momentum have no explicit time dependence. To evaluate their Poisson brackets with  $\tilde{P}_0$ , we shall use the following procedure. Our improved generators are (non-local) functionals, having the form of integrals of some local densities. The Poisson bracket of two such generators can be calculated by acting with one of the generators on the integrand of the other. In the case of linear momentum, we have

$$\{\tilde{P}_0, \tilde{P}_\alpha\} = \int d^3x \{\hat{\mathcal{H}}_T + \partial_\gamma \pi_0^\gamma, \tilde{P}_\alpha\} \approx \int d^3x \partial_\gamma \{\pi_0^\gamma, \tilde{P}_\alpha\},$$

because  $\hat{\mathcal{H}}_T \approx 0$  is a constraint in the theory, and therefore, weakly commutes with all the symmetry generators. The last term in the above formula is easily evaluated (Appendix C), with the final result

$$\{\tilde{P}_0, \tilde{P}_\alpha\} \approx \oint dS_\gamma \partial_\alpha \pi_0^\gamma = 0, \quad (4.5)$$

as a consequence of  $\partial_\alpha \pi_0^\gamma = \mathcal{O}_3$ . Therefore, no surface term appears in (4.3) for  $G = \tilde{P}_\alpha$ , and we have the conservation law

$$\frac{d\tilde{P}_\alpha}{dt} \approx \frac{dE_\alpha}{dt} \approx 0. \quad (4.6)$$

3. In a similar way, we can check the conservation of the rotation generator. Using the results of Appendix C, we find:

$$\begin{aligned} \{\tilde{P}_0, \tilde{M}_{\alpha\beta}\} &\approx \int d^3x \partial_\gamma \{\pi_0^\gamma, \tilde{M}_{\alpha\beta}\} \approx \int d^3x (x_\alpha \partial_\beta - x_\beta \partial_\alpha) \partial_\gamma \pi_0^\gamma \\ &\approx \int d^3x (\partial_\beta x_\alpha - \partial_\alpha x_\beta) \partial_\gamma \pi_0^\gamma \approx \oint (x_\alpha dS_\beta - x_\beta dS_\alpha) \partial_\gamma \pi_0^\gamma = 0, \end{aligned}$$

because  $\partial_\gamma \pi_0^\gamma = \mathcal{O}_4$  according to (3.18). Therefore, the surface term in (4.3) is absent for  $G = \tilde{M}_{\alpha\beta}$ , and the on-shell value of the rotation generator is conserved:

$$\frac{d\tilde{M}_{\alpha\beta}}{dt} \approx \frac{dE_{\alpha\beta}}{dt} \approx 0. \quad (4.7)$$

4. Finally, the boost generator (2.2b) has an explicit, linear dependence on time, and satisfies

$$\{\tilde{M}_{0\beta}, \tilde{P}_0\} + \frac{\partial \tilde{M}_{0\beta}}{\partial t} = \{\tilde{M}_{0\beta}, \tilde{P}_0\} + \tilde{P}_\beta \approx -\{\tilde{P}_0, \tilde{M}_{0\beta}\} + E_\beta. \quad (4.8)$$

The evaluation of the Poisson bracket in (4.8) is done with the help of Appendix C:

$$\{\tilde{P}_0, \tilde{M}_{0\beta}\} \approx \int d^3x \partial_\gamma \left[ \pi_\beta{}^\gamma - \partial_\alpha \left( x_\beta \frac{\partial \mathcal{L}}{\partial b^0{}_{\alpha,\gamma}} \right) \right] = \int d^3x \partial_\gamma \pi_\beta{}^\gamma = E_\beta, \quad (4.9)$$

where we used the antisymmetry of  $\partial \mathcal{L} / \partial b^0{}_{\alpha,\gamma}$  in  $(\alpha, \gamma)$ , and  $\partial_\gamma \pi_0{}^\gamma = \mathcal{O}_4$ . As opposed to all the other generators, the Poisson bracket of the boost generator with  $\tilde{P}_0$  does not vanish: its on-shell value is precisely the value of the linear momentum  $E_\beta$ . Substitution of this result back into (4.8) yields the boost conservation law [36]:

$$\frac{d\tilde{M}_{0\beta}}{dt} \approx \frac{dE_{0\beta}}{dt} \approx 0. \quad (4.10)$$

In conclusion, *all ten Poincaré generators of the general teleparallel theory are conserved quantities* in the phase space defined by the appropriate asymptotic and parity conditions.

## V. LAGRANGIAN FORM OF THE CONSERVED CHARGES

In this section, we wish to transform our Hamiltonian expressions for the conserved quantities into the Lagrangian form, and compare the obtained results with the related GR expressions. This will be achieved by expressing all momentum variables, in  $E_\mu$  and  $E_{\mu\nu}$ , in terms of the fields and their derivatives, using the defining relations  $\pi_A = \partial \mathcal{L} / \partial \dot{\varphi}^A$ . All calculations refer to the one-parameter teleparallel theory ( $2A + B = 1$ ,  $C = -1$ ).

### A. Energy and momentum

The energy-momentum expression (3.8) can be transformed into the Lagrangian form with the help of the relation (D3) that defines  $\pi_i{}^\gamma$ :

$$E_\mu = \oint dS_\gamma H_\mu{}^{0\gamma}, \quad H_\mu{}^{0\gamma} \equiv -4bb^i{}_\mu h^{j\gamma} \beta_{ij}{}^0. \quad (5.1)$$

Now, we wish to compare this result with GR.

Using the decomposition of  $f^i{}_\mu \equiv b^i{}_\mu - \delta^i{}_\mu$  into symmetric and antisymmetric part,  $f_{i\mu} = s_{i\mu} + a_{i\mu}$ , and the asymptotic conditions (2.3)–(2.5), we obtain

$$H_0{}^{0\gamma} = 2a(s_c{}^{c,\gamma} - s_c{}^{\gamma,c}) - 2aa^{c\gamma}{}_{,c} + \mathcal{O}_3.$$

Note that the second term in  $\beta_{ij}{}^0$ , proportional to  $(2B - 1)$  and given by Eq. (D4), does not contribute to this result. Now, after introducing the well known superpotential of Landau and Lifshitz,

$$h^{\mu\nu\lambda} = \partial_\rho \psi^{\mu\nu\lambda\rho}, \quad \psi^{\mu\nu\lambda\rho} \equiv a(-g)(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho}),$$

one easily verifies that  $h^{00\gamma} = 2a(s_c^{c,\gamma} - s_c^{\gamma,c}) + \mathcal{O}_3$ . Then, after discarding the inessential divergence of the antisymmetric tensor in  $H_0^{0\gamma}$ , we find the following Lagrangian expression for  $E_0$ :

$$E_0 = \oint dS_\gamma h^{00\gamma}. \quad (5.2)$$

Thus, the energy of the one-parameter teleparallel theory is given by the *same* expression as in GR.

In a similar manner, we can transform the expression for  $E_\alpha$ . Starting with equation (D4), we note that the first term in  $\beta_{ij}^0$ , which corresponds to  $\text{GR}_\parallel$  ( $B = 1/2$ ), gives the contribution

$$\begin{aligned} (H_\alpha^{0\gamma})_{(1)} &= 2a[\eta^{\gamma\beta}(s_{\alpha\beta,0} - s_{0\alpha,\beta}) + \delta_\alpha^\gamma(s_{0,c}^c - s_{c,0}^c)] \\ &\quad + 4a(\delta_\alpha^{[\gamma} a^{\beta]}_0)_{,\beta} + \mathcal{O}_3, \end{aligned}$$

which, after dropping the irrelevant divergence of the antisymmetric tensor, can be identified with  $h_\alpha^{0\gamma}$ . The contribution of the second term has the form

$$\begin{aligned} (H_\alpha^{0\gamma})_{(2)} &= a(2B - 1)bb^i_\alpha h^{j\gamma} h^{k0} \overset{A}{T}_{ijk} \\ &= a(2B - 1) \overset{A}{T}_\alpha^{\gamma 0} + \mathcal{O}_3, \end{aligned}$$

where  $\overset{A}{T}_{ijk} = T_{ijk} + T_{kij} + T_{jki}$ . Thus, the complete linear momentum takes the form

$$E_\alpha = \eta_{\alpha\mu} \oint dS_\gamma \left[ h^{\mu 0\gamma} - a(2B - 1) \overset{A}{T}^{\mu 0\gamma} \right], \quad (5.3)$$

which is *different* from what we have in GR.

Energy and momentum expressions (5.2) and (5.3) can be written in a Lorentz covariant form as

$$\begin{aligned} E^\mu &= \oint dS_\gamma \bar{h}^{\mu 0\gamma} = \int d^3x \bar{\theta}^{\mu 0}, \\ \bar{\theta}^{\mu\nu} &\equiv \bar{h}^{\mu\nu\rho}{}_{,\rho}, \quad \bar{h}^{\mu\nu\rho} \equiv h^{\mu\nu\rho} - a(2B - 1) \overset{A}{T}^{\mu\nu\rho}. \end{aligned} \quad (5.4)$$

In the case  $2B - 1 = 0$ , corresponding to  $\text{GR}_\parallel$ , we see that  $\bar{\theta}^{\mu\nu}$  coincides with the Landau-Lifshitz symmetric energy-momentum complex  $\theta^{\mu\nu} \equiv h^{\mu\nu\rho}{}_{,\rho}$  of GR. When  $2B - 1 \neq 0$ , the momentum acquires a *correction* proportional to the totally antisymmetric part of the torsion.

## B. Angular momentum

The elimination of momenta from Eq. (3.19) leads to

$$E_{\mu\nu} = \oint dS_\gamma (x_\mu H_\nu^{0\gamma} - x_\nu H_\mu^{0\gamma} - 4\lambda_{\mu\nu}^{0\gamma}), \quad (5.5)$$

where we used the expression (D3) for  $\pi_k^\alpha$ , and  $\pi_{\mu\nu}^\gamma \approx 4\delta_\mu^i \delta_\nu^j \lambda_{ij}^{0\gamma} \equiv 4\lambda_{\mu\nu}^{0\gamma}$ .

In order to compare this result with GR, we shall first eliminate  $\lambda$  using the second field equation (1.3b). This is most easily done in the gauge  $A_{\mu}^{ij} = 0$ . In what follows, all the calculations and results refer to this gauge. Thus, Eq. (1.3b) in the gauge fixed form gives:

$$4\partial_{\gamma}\lambda_{ij}{}^{0\gamma} \approx 8b\beta_{[ij]}^0 + \sigma^0_{ij}. \quad (5.6)$$

The first term in  $\beta_{ij}^0$  is given in Eq. (D4), and corresponds to  $\text{GR}_{\parallel}$  ( $2B - 1 = 0$ ); in particular,  $4b(\beta_{[ij]}^0)_{(1)} \approx -a\partial_{\gamma}H_{ij}^{0\gamma}$ . Then, we define  $\lambda_{(1)}$  by ignoring the term  $\sigma^0_{ij}$  in (5.6):

$$\partial_{\gamma}\left[4(\lambda_{ij}{}^{0\gamma})_{(1)} + 2aH_{ij}^{0\gamma}\right] \approx 0.$$

The related contribution to the angular momentum has the form (see Appendix D):

$$E_{(1)}^{\mu\nu} = \oint dS_{\gamma}[x^{\mu}h^{\nu 0\gamma} - x^{\nu}h^{\mu 0\gamma} + \psi^{\mu 0\gamma\nu} + \mathcal{O}(f^2)], \quad (5.7a)$$

where  $\psi^{\mu 0\gamma\nu}$  satisfies the relation  $\partial_{\gamma}\psi^{\mu 0\gamma\nu} = h^{\mu 0\nu} - h^{\nu 0\mu}$ , and  $\mathcal{O}(f^2)$  denotes terms quadratic in  $f^i_{\mu}$  and/or its derivatives. The parity conditions (3.16) ensure that the  $\mathcal{O}(f^2)$  terms do not contribute to the surface integral, hence

$$E_{(1)}^{\mu\nu} = \oint dS_{\gamma}K^{\mu\nu 0\gamma} = \int d^3x(x^{\mu}\theta^{\nu 0} - x^{\nu}\theta^{\mu 0}). \quad (5.7b)$$

Here, the tensor  $K^{\mu\nu\lambda\rho} = x^{\mu}h^{\nu\lambda\rho} - x^{\nu}h^{\mu\lambda\rho} + \psi^{\lambda\mu\nu\rho}$  satisfies the relation

$$\partial_{\rho}K^{\mu\nu\lambda\rho} = x^{\mu}\theta^{\nu\lambda} - x^{\nu}\theta^{\mu\lambda},$$

and  $E_{(1)}^{\mu\nu}$  represents the angular momentum of GR, as expected.

The second term of  $\beta_{ij}^0$  in Eq. (D4) defines the second term of  $\lambda = \lambda_{(1)} + \lambda_{(2)}$ :

$$4\partial_{\gamma}(\lambda_{ij}{}^{0\gamma})_{(2)} \approx -2a(2B - 1)bh^{k0} \overset{A}{T}_{ijk} + \sigma^0_{ij}.$$

The related angular momentum contribution takes the form

$$E_{(2)}^{\mu\nu} = \oint dS_{\gamma}\left[a(2B - 1)(x^{\mu}\overset{A}{T}{}^{\nu\gamma 0} - x^{\nu}\overset{A}{T}{}^{\mu\gamma 0}) - 4\eta^{i\mu}\eta^{j\nu}(\lambda_{ij}{}^{0\gamma})_{(2)}\right], \quad (5.8)$$

where the inessential  $\mathcal{O}(f^2)$  terms are ignored.

The complete angular momentum can be transformed into the form of a volume integral:

$$\begin{aligned} E^{\mu\nu} = \int d^3x \Big\{ & (x^{\mu}\bar{\theta}^{\nu 0} - x^{\nu}\bar{\theta}^{\mu 0}) + 2a(2B - 1)\overset{A}{T}{}^{\nu\mu 0} \\ & + \eta^{i\mu}\eta^{j\nu}\left[2a(2B - 1)bh^{k0} \overset{A}{T}_{ijk} - \sigma^0_{ij}\right] \Big\}. \end{aligned} \quad (5.9)$$

Note that the integrand of  $E^{\mu\nu}$  is not of the simple form  $x^{\mu}\bar{\theta}^{\nu 0} - x^{\nu}\bar{\theta}^{\mu 0}$ , since the corrected energy-momentum complex  $\bar{\theta}^{\mu\nu}$  is (in general) not symmetric. We see that even in  $\text{GR}_{\parallel}$  ( $2B - 1 = 0$ ), the angular momentum differs from the GR expression (by the contribution of the  $\sigma^0_{ij}$  term).

Our results for the conserved charges are valid for any solution satisfying the asymptotic conditions defined by equations (2.3)–(2.5) and (3.16). Assuming that the gravitational field is produced by the Dirac field as a source, Hayashi and Shirafuji [9] found a specific solution for which  $f_{[0\alpha]} = k\varepsilon_{\alpha\beta\gamma}n^\beta S^\gamma/r^2$ , where  $S^\gamma$  is the spin of the source, and  $k$  is a constant. The solution is obtained in the weak field approximation, and yields a non-vanishing antisymmetric part of the torsion. Using our expressions for the conserved charges, we verified that this solution gives a vanishing correction to the GR results. This is in agreement with the results of Kawai and Toma [37], who studied the Noether charges in a non-standard formulation of the teleparallel theory, and concluded that all contributions stemming from  $\overset{A}{T}_{ijk}$  effectively vanish.

## VI. CONCLUDING REMARKS

In this paper, we presented an investigation of the connection between the asymptotic Poincaré symmetry of spacetime and the related conservation laws of energy-momentum and angular momentum in the general teleparallel theory of gravity.

The generators of the global Poincaré symmetry in the asymptotic region are derived from the related gauge generators, constructed in Ref. [27]. Since these generators act on dynamical variables via the Poisson bracket operation, it is natural to demand that they have well defined functional derivatives in a properly defined phase space. This requirement leads to the conclusion that the Poincaré generators have to be improved by adding certain surface terms, which represent the values of energy-momentum and angular momentum of the gravitating system.

The general asymptotic behavior of dynamical variables, determined by Eqs. (2.3)–(2.5), is sufficient to guarantee the existence of well behaved (finite and differentiable) generators of spacetime translations and spatial rotations. This is, however, not true for boosts: they can be improved by adding surface integrals only if one imposes the additional parity conditions (3.16). Using the canonical criterion (4.2), we were able to show that the improved generators are not only finite, but also conserved; hence, the related charges, energy-momentum and angular momentum, are the constants of motion.

Our results for energy-momentum and angular momentum are valid for the *general teleparallel theory*. In the context of GR<sub>||</sub>, Nester [38] used some geometric arguments to derive an energy-momentum expression, which agrees with our formula (3.8). On the basis of this result, he was able to formulate a pure tensorial proof of the positivity of energy in GR, in terms of the teleparallel geometry. Using the constraint  $\pi_{\perp\bar{k}} + 2aJT^{\bar{m}}_{\bar{m}\bar{k}} \approx 0$ , which holds in GR<sub>||</sub> (Appendix A), one can derive another equivalent form of the energy integral,  $E_0 = -2a \int d^3x \partial_\gamma T_a^{\alpha\gamma}$ , appearing in the literature [39].

After transforming the obtained surface integrals to the Lagrangian form, one finds that the conserved charges in the one-parameter teleparallel theory *differ* (in general) from the corresponding GR expressions. Mielke and Wallner [40] discussed Lagrangian form of the energy-momentum of some exact solutions in the teleparallel limit of PGT. A purely Lagrangian analysis of the energy-momentum in the specific case of GR<sub>||</sub> without matter has been carried out in Ref. [41], with the same result as in GR. Kawai and Toma [37] studied Noether charges for both energy-momentum and angular momentum, in a non-

standard formulation of the teleparallel theory, and found that these charges have the *same* form as in GR. Such a conclusion can be understood by noting that, effectively, they used specific boundary conditions corresponding to the Hayashi–Shirafuji solution [9], for which the conserved charges are indeed of the same form as in GR.

The results obtained in this paper can be used to justify some proposals used in the literature for the energy-momentum in GR<sub>||</sub> [37–39,41], extend the concepts of energy-momentum and angular momentum from GR<sub>||</sub> to the general teleparallel theory, and study the related stability properties.

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## APPENDIX A: HAMILTONIAN AND GAUGE GENERATORS

The canonical dynamics of the general teleparallel theory (1.2) is described by the total Hamiltonian [17]

$$\begin{aligned}\mathcal{H}_T &= \hat{\mathcal{H}}_T + \partial_\alpha \bar{D}^\alpha, \\ \hat{\mathcal{H}}_T &\equiv \bar{\mathcal{H}}_c + u^i{}_0 \pi_i{}^0 + \frac{1}{2} u^{ij}{}_0 \pi_{ij}{}^0 + \frac{1}{4} u_{ij}{}^{\alpha\beta} \pi^{ij}{}_{\alpha\beta} + (u \cdot \phi), \\ \bar{D}^\alpha &= b^k{}_0 \pi_k{}^\alpha + \frac{1}{2} A^{ij}{}_0 \pi_{ij}{}^\alpha - \frac{1}{2} \lambda_{ij}{}^{\alpha\beta} \pi^{ij}{}_{0\beta},\end{aligned}\tag{A1a}$$

where  $\pi_i{}^0$ ,  $\pi_{ij}{}^0$ ,  $\pi^{ij}{}_{\alpha\beta}$  and  $\phi$  are primary constraints,  $u$  are the corresponding multipliers, and  $\bar{\mathcal{H}}_c$  is the canonical Hamiltonian,

$$\bar{\mathcal{H}}_c = N \bar{\mathcal{H}}_\perp + N^\alpha \bar{\mathcal{H}}_\alpha - \frac{1}{2} A^{ij}{}_0 \bar{\mathcal{H}}_{ij} - \lambda_{ij}{}^{\alpha\beta} \bar{\mathcal{H}}^{ij}{}_{\alpha\beta},\tag{A1b}$$

whose components are given by

$$\begin{aligned}\bar{\mathcal{H}}_\alpha &= \pi_i{}^\beta T_{\alpha\beta}^i - b^k{}_\alpha \nabla_\beta \pi_k{}^\beta + \frac{1}{2} \pi^{ij}{}_{0\alpha} \nabla_\beta \lambda_{ij}{}^{0\beta}, \\ \bar{\mathcal{H}}_{ij} &= 2\pi_{[i}{}^\alpha b_{j]\alpha} + \nabla_\alpha \pi_{ij}{}^\alpha + 2\pi^s{}_{[i0\alpha} \lambda_{sj]}{}^{0\alpha}, \\ \bar{\mathcal{H}}^{ij}{}_{\alpha\beta} &= R^{ij}{}_{\alpha\beta} - \frac{1}{2} \nabla_{[\alpha} \pi^{ij}{}_{0\beta]}, \\ \bar{\mathcal{H}}_\perp &= \mathcal{H}_\perp - \frac{1}{8} (\partial \mathcal{H}_\perp / \partial A^{ij}{}_\alpha) \pi^{ij}{}_{0\alpha}, \\ \mathcal{H}_\perp &\equiv \hat{\pi}_i{}^{\bar{k}} T_{\perp \bar{k}}^i - J \mathcal{L}_T - n^k \nabla_\beta \pi_k{}^\beta.\end{aligned}\tag{A1c}$$

Here,  $\nabla_\mu$  is the covariant derivative,  $n_k = h_k{}^0 / \sqrt{g^{00}}$  is the unit normal to the hypersurface  $x^0 = \text{const}$ , the bar over the Latin index is defined by the decomposition  $V_k = V_\perp n_k + V_{\bar{k}}$ ,  $V_\perp = n^k V_k$ , of an arbitrary vector  $V_k$ , the lapse and shift functions  $N$  and  $N^\alpha$  are given as  $N = n_k b^k{}_0$ ,  $N^\alpha = h_{\bar{k}}{}^\alpha b^k{}_0$ ,  $J$  is determined by  $b = NJ$ , and  $\hat{\pi}_{i\bar{k}} = \pi_i{}^\alpha b_{k\alpha}$ . Note that  $\bar{\mathcal{H}}_c$  is linear in unphysical variables ( $b^k{}_0, A^{ij}{}_0, \lambda_{ij}{}^{\alpha\beta}$ ).

Table 1.

	first class	second class
primary	$\pi_i{}^0, \pi_{ij}{}^0, \pi^{ij}{}_{\alpha\beta}$	$\phi_{ij}{}^\alpha, \pi^{ij}{}_{0\beta}$
secondary	$\bar{\mathcal{H}}_\perp, \bar{\mathcal{H}}_\alpha, \bar{\mathcal{H}}_{ij}, \bar{\mathcal{H}}^{ij}{}_{\alpha\beta}$	

The dynamical classification of the sure constraints is given in Table 1, where

$$\phi_{ij}{}^\alpha = \pi_{ij}{}^\alpha - 4\lambda_{ij}{}^{0\alpha}. \quad (\text{A2})$$

The only terms in the total Hamiltonian  $\mathcal{H}_T$  that depend on the specific form of the Lagrangian are dynamical Hamiltonian  $\mathcal{H}_\perp$  and extra primary constraints  $\phi$ . We display here, for completeness, the explicit form of these quantities for the teleparallel formulation of GR (GR<sub>||</sub>) [17]:

$$\begin{aligned} \mathcal{H}_\perp &= \frac{1}{2}P_T^2 - J\mathcal{L}_T(\bar{T}) - n^k \nabla_\beta \pi_k{}^\beta, \\ P_T^2 &= \frac{1}{2aJ} \left( \hat{\pi}_{(\bar{i}\bar{k})} \hat{\pi}^{(\bar{i}\bar{k})} - \frac{1}{2} \hat{\pi}^{\bar{m}}{}_{\bar{m}} \hat{\pi}^{\bar{n}}{}_{\bar{n}} \right), \\ \mathcal{L}_T(\bar{T}) &= a \left( \frac{1}{4} T_{\bar{m}\bar{n}\bar{k}} T^{m\bar{n}\bar{k}} + \frac{1}{2} T_{\bar{m}\bar{n}\bar{k}} T^{\bar{n}\bar{m}\bar{k}} - T^{\bar{m}}{}_{\bar{m}\bar{k}} T_{\bar{n}}{}^{\bar{n}\bar{k}} \right), \end{aligned} \quad (\text{A3a})$$

and

$$\begin{aligned} (u \cdot \phi) &= \frac{1}{2} u^{ik} \bar{\phi}_{ik}, \\ \bar{\phi}_{ik} &= \phi_{ik} - \frac{1}{4} a (\pi_i{}^s{}_{0\alpha} B_{sk}^{0\alpha} + \pi_k{}^s{}_{0\alpha} B_{is}^{0\alpha}), \\ \phi_{ik} &= \hat{\pi}_{i\bar{k}} - \hat{\pi}_{k\bar{i}} + a \nabla_\alpha B_{ik}^{0\alpha}, \end{aligned} \quad (\text{A3b})$$

where  $B_{ik}^{0\alpha} \equiv \varepsilon^{0\alpha\beta\gamma} \varepsilon_{ikmn} b_\beta^m b_\gamma^n$ .

The Poincaré gauge generator of the general teleparallel theory (1.2) has the form [27]

$$G = G(\omega) + G(\xi), \quad (\text{A4a})$$

where

$$\begin{aligned} G(\omega) &= -\frac{1}{2} \dot{\omega}^{ij} \pi_{ij}{}^0 - \frac{1}{2} \omega^{ij} S_{ij}, \\ G(\xi) &= -\dot{\xi}^0 (b^k{}_0 \pi_k{}^0 + \frac{1}{2} A^{ij}{}_0 \pi_{ij}{}^0 + \frac{1}{4} \lambda_{ij}{}^{\alpha\beta} \pi_{\alpha\beta}{}^{ij}) - \xi^0 \mathcal{P}_0 \\ &\quad - \dot{\xi}^\alpha (b^k{}_\alpha \pi_k{}^0 + \frac{1}{2} A^{ij}{}_\alpha \pi_{ij}{}^0 - \frac{1}{2} \lambda_{ij}{}^{0\beta} \pi_{\alpha\beta}{}^{ij}) - \xi^\alpha \mathcal{P}_\alpha. \end{aligned} \quad (\text{A4b})$$

In the above expressions, we used the following notation:

$$\begin{aligned} S_{ij} &= -\bar{\mathcal{H}}_{ij} + 2b_{[i0}\pi_{j]}{}^0 + 2A^s{}_{[i0}\pi_{s]j}{}^0 + 2\lambda_{s[i}{}^{\alpha\beta} \pi_{j]\alpha\beta}{}^s, \\ \mathcal{P}_0 &\equiv \hat{\mathcal{H}}_T = \mathcal{H}_T - \partial_\alpha \bar{D}^\alpha, \\ \mathcal{P}_\alpha &= \bar{\mathcal{H}}_\alpha - \frac{1}{2} A^{ij}{}_\alpha \bar{\mathcal{H}}_{ij} + 2\lambda_{ij}{}^{0\beta} \bar{\mathcal{H}}_{\alpha\beta}{}^{ij} + \pi_k{}^0 \partial_\alpha b^k{}_0 + \frac{1}{2} \pi_{ij}{}^0 \partial_\alpha A^{ij}{}_0 \\ &\quad - \frac{1}{4} \lambda_{ij}{}^{\beta\gamma} \partial_\alpha \pi_{\beta\gamma}{}^{ij} - \frac{1}{2} \partial_\gamma (\lambda_{ij}{}^{\beta\gamma} \pi_{\alpha\beta}{}^{ij}). \end{aligned} \quad (\text{A4c})$$

## APPENDIX B: ON SURFACE TERMS

In this Appendix, we shall discuss asymptotic properties of vector fields, which are important for understanding the structure of surface terms.

Consider a vector field  $\mathbf{A}$ , with the following asymptotic behavior:

$$\mathbf{A} = \frac{\mathbf{a}}{r} + \mathcal{O}_2, \quad \text{div } \mathbf{A} = \mathcal{O}_3, \quad (\text{B1})$$

where  $\mathbf{a} = \mathbf{a}(\mathbf{n})$ ,  $\mathbf{n} = \mathbf{x}/r$ . We shall prove that under these assumptions

$$\oint_{S_\infty} \mathbf{A} \cdot d\mathbf{S} = \text{finite}, \quad (\text{B2})$$

where  $S_\infty$  is the sphere at spatial infinity.

Relations (B1) imply  $\text{div}(\mathbf{a}/r) = 0$ , for all  $r \neq 0$ . Integrating  $\text{div}(\mathbf{a}/r)$  over the region  $V$  outside the sphere  $S_R$  of radius  $R$ , and using Gauss divergence theorem yields

$$\oint_{S_\infty} \frac{\mathbf{a}}{r} \cdot d\mathbf{S} - \oint_{S_R} \frac{\mathbf{a}}{r} \cdot d\mathbf{S} = 0, \quad \forall R \neq 0.$$

Here, the integral over  $S_\infty$  is independent of  $R$ , while the integral over  $S_R$  is linear in  $R$ :

$$\oint_{S_R} \frac{\mathbf{a}}{r} \cdot d\mathbf{S} = R \oint \mathbf{a} \cdot \mathbf{n} d\Omega.$$

Hence, we must have  $\oint \mathbf{a} \cdot \mathbf{n} d\Omega = 0$ , which is equivalent to

$$\oint_{S_\infty} \frac{\mathbf{a}}{r} \cdot d\mathbf{S} = 0.$$

Now, using the first relation in (B1) we easily verify the statement (B2).

Note that the same line of reasoning in the case of

$$\mathbf{B} = \frac{\mathbf{b}}{r^2} + \mathcal{O}_3, \quad \text{div } \mathbf{B} = \mathcal{O}_4, \quad (\text{B3})$$

does not lead to  $\oint \mathbf{B} \cdot d\mathbf{S} = 0$ , as one might naively expect. For example, the everywhere regular vector field

$$\mathbf{B} = \frac{1 + \sqrt{1 + r^2}}{(1 + r^2)^2} \mathbf{x}$$

satisfies (B3), but yields  $\oint \mathbf{B} \cdot d\mathbf{S} = 4\pi$ .

## APPENDIX C: CONSERVATION LAWS

To verify the conservation of the improved Poincaré generators, we need their Poisson brackets with  $\tilde{P}_0$ . The essential part of these brackets is the expression  $\{\pi_0^\gamma, G\}$ , representing the action of the Poincaré generators on the local quantity  $\pi_0^\gamma$ . Using the known Poincaré transformation law for the momenta, as given by the equation (5.2) of Ref. [27], we can write:

$$\delta_0 \pi_k^\gamma = \{\pi_k^\gamma, G\} \approx \varepsilon_k^s \pi_s^\gamma + \varepsilon^\gamma_\beta \pi_k^\beta + \varepsilon^0_\beta \frac{\partial \mathcal{L}}{\partial b^k_{\gamma, \beta}} - (\varepsilon^\mu_\nu x^\nu + \varepsilon^\mu) \partial_\mu \pi_k^\gamma, \quad (\text{C1})$$

wherefrom we read the corresponding Poisson brackets. In the case of  $\pi_0^\gamma$ , we have:

$$\begin{aligned} \{\pi_0^\gamma, \tilde{P}_\mu\} &\approx \partial_\mu \pi_0^\gamma, \\ \{\pi_0^\gamma, \tilde{M}_{\alpha\beta}\} &\approx (\delta_\alpha^\gamma \pi_{0\beta} + x_\alpha \partial_\beta \pi_0^\gamma) - (\alpha \leftrightarrow \beta), \\ \{\pi_0^\gamma, \tilde{M}_{0\beta}\} &\approx \pi_\beta^\gamma + \eta_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial b^0_{\gamma, \alpha}} + x_0 \partial_\beta \pi_0^\gamma - x_\beta \partial_0 \pi_0^\gamma. \end{aligned} \quad (\text{C2})$$

The last of the above equations can further be simplified by using the Lagrangian equations of motion. Thus,

$$\partial_0 \pi_0^\gamma \approx \partial_0 \frac{\partial \mathcal{L}}{\partial b_{\gamma,0}^0} \approx \partial_\alpha \frac{\partial \mathcal{L}}{\partial b_{\alpha,\gamma}^0} + \frac{\partial \mathcal{L}}{\partial b_\gamma^0},$$

so that the  $x_\beta \partial_0 \pi_0^\gamma$  part becomes

$$x_\beta \partial_0 \pi_0^\gamma = x_\beta \partial_\alpha \frac{\partial \mathcal{L}}{\partial b_{\alpha,\gamma}^0} + \mathcal{O}_3 = \partial_\alpha \left( x_\beta \frac{\partial \mathcal{L}}{\partial b_{\alpha,\gamma}^0} \right) + \eta_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial b_{\gamma,\alpha}^0} + \mathcal{O}_3. \quad (\text{C3})$$

Its substitution back into (C2) gives the result used in equation (4.9).

## APPENDIX D: CONNECTION WITH THE LAGRANGIAN FORMALISM

In this Appendix we collect some formulas which simplify the derivation of the Lagrangian form of the conserved charges.

1. We begin with the relation

$$\begin{aligned} \beta_{ijk} &= \frac{1}{2} a (\tau_{jki} - \tau_{ijk} + \tau_{kij}) - \frac{1}{4} a (2B - 1) \overset{A}{T}_{ijk}, \\ \tau_{kij} &= \eta_{k[i} T_{j]} - \frac{1}{2} T_{kij}. \end{aligned} \quad (\text{D1a})$$

After using the identity  $2b\tau_{kij} = -b_{k\lambda} \nabla_\rho H_{ij}^{\lambda\rho}$ , where  $H_{ij}^{\mu\nu} \equiv b(h_i^\mu h_j^\nu - h_j^\mu h_i^\nu)$ , one easily obtains

$$\begin{aligned} 4b\beta_{(ij)}^\mu &= ah^{k\mu} b_{i\lambda} \nabla_\rho H_{jk}^{\lambda\rho} + (i \leftrightarrow j), \\ 4b\beta_{[ij]}^\mu &= -a \nabla_\rho H_{ij}^{\mu\rho} - a(2B - 1) bh^{k\mu} \overset{A}{T}_{ijk}. \end{aligned} \quad (\text{D1b})$$

When we combine the last equation with the weak relation  $\nabla_\mu (4b\beta_{[ij]}^\mu + \sigma_{ij}^\mu/2) \approx 0$ , which follows from the equations of motion (1.3), we obtain the relation

$$\nabla_\mu [2a(2B - 1) bh^{k\mu} \overset{A}{T}_{ijk} - \sigma_{ij}^\mu] \approx 0, \quad (\text{D2a})$$

which, in the weak field approximation, reads

$$\partial_\rho [2a(2B - 1) \overset{A}{T}_{\mu\nu}{}^\rho - \sigma_{\mu\nu}^\rho] \approx \mathcal{O}(f^2) + \hat{\mathcal{O}} = \mathcal{O}_4, \quad (\text{D2b})$$

where  $\mathcal{O}(f^2)$  denotes terms quadratic in  $f_\mu^i$  and/or its derivatives.

2. The momentum  $\pi_i^\gamma$  is defined by the relation

$$\pi_i^\gamma = -4bh^{j\gamma} \beta_{ij}^0. \quad (\text{D3})$$

In order to simplify the calculations, we rewrite  $\beta_{ij}^0$  as a sum of two terms:

$$\begin{aligned} \beta_{ij}^0 &= (\beta_{ij}^0)_{(1)} + (\beta_{ij}^0)_{(2)}, \\ 4b(\beta_{ij}^0)_{(1)} &= ah^{k0} (b_{i\lambda} \nabla_\rho H_{jk}^{\lambda\rho} + b_{j\lambda} \nabla_\rho H_{ik}^{\lambda\rho}) - a \nabla_\rho H_{ij}^{0\rho}, \\ 4b(\beta_{ij}^0)_{(2)} &= -a(2B - 1) bh^{k0} \overset{A}{T}_{ijk}. \end{aligned} \quad (\text{D4})$$

The first term corresponds to  $\text{GR}_{\parallel}$  ( $2B - 1 = 0$ ), and its contribution to the angular momentum  $E_{\mu\nu} = \int d^3x \mathcal{E}_{\mu\nu}$  has the form

$$\mathcal{E}_{\mu\nu}^{(1)} = \partial_\gamma \left[ a \delta_\mu^i \delta_\nu^j H_{ij}^{0\gamma} - 4b x_\mu b_\nu^i h^{j\gamma} (\beta_{ij}^0)_{(1)} \right] - (\mu \leftrightarrow \nu) + \hat{\mathcal{O}}. \quad (\text{D5a})$$

Going now back to the surface integral, we can use the parity conditions (3.16) to conclude that the terms quadratic in  $f_\mu^i$  and of order  $r^{-2}$  are even, so that their contribution to the surface integral vanishes. Hence, we continue by keeping only linear terms:

$$\begin{aligned} \mathcal{E}_{(1)}^{\mu\nu} &= a \partial_\gamma \left[ H^{\mu\nu 0\gamma} + x^\mu (H^{\nu\gamma 0\rho} - H^{\gamma 0\nu\rho} - H^{\nu 0\gamma\rho})_{,\rho} \right] \\ &\quad - (\mu \leftrightarrow \nu) + \partial \mathcal{O}(f^2) + \hat{\mathcal{O}}, \end{aligned} \quad (\text{D5b})$$

where  $H^{ij\mu\nu} = \eta^{im} \eta^{jn} H_{mn}^{\mu\nu}$ . Using the decomposition  $H^{ij\mu\nu} = H_S^{ij\mu\nu} + H_A^{ij\mu\nu}$ , where  $H_S^{ij\mu\nu}$  is symmetric and  $H_A^{ij\mu\nu}$  antisymmetric under the exchange of the pairs of indices  $(ij) \leftrightarrow (\mu\nu)$ , and the identities

$$\begin{aligned} \psi^{\mu\nu\lambda\rho} &= ab h^{i\nu} h^{j\lambda} H_{ij}^{\mu\rho} = 2a (H_S^{\mu\rho\nu\lambda} - \eta^{[\mu\nu} \eta^{\rho]\lambda}) + \mathcal{O}(f^2), \\ \psi^{ij\mu\nu} - \psi^{\mu j i \nu} &= \psi^{ij\nu\mu}, \end{aligned}$$

we can derive the relation

$$a (H_S^{\nu\gamma 0\rho} - H_S^{\gamma 0\nu\rho} - H_S^{\nu 0\gamma\rho})_{,\rho} = \psi^{\gamma\rho\nu 0}_{,\rho} = h^{\nu 0\gamma}.$$

After substituting  $H \rightarrow H_S$  and  $H \rightarrow H_A$ , respectively, in Eq. (D5b), we find

$$\begin{aligned} \mathcal{E}_{(1)}^{\mu\nu}(H_S) &= \partial_\gamma (\psi^{\mu 0\gamma\nu} + x^\mu h^{\nu 0\gamma} - x^\nu h^{\mu 0\gamma}) + \partial \mathcal{O}(f^2) + \hat{\mathcal{O}}, \\ \mathcal{E}_{(1)}^{\mu\nu}(H_A) &= \partial_\gamma \partial_\beta \left[ a x^\mu (H_A^{0\nu\gamma\beta} + 2H_A^{0[\gamma\nu\beta]}) - (\mu \leftrightarrow \nu) \right] + \partial \mathcal{O}(f^2) + \hat{\mathcal{O}}. \end{aligned} \quad (\text{D6})$$

The term  $\mathcal{E}_{(1)}^{\mu\nu}(H_A)$  effectively vanishes, since the expression in square brackets is antisymmetric in  $(\gamma, \beta)$ , and the final expression for the angular momentum  $E_{(1)}^{\mu\nu}$  has the form (5.7a).

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